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A short remark on Horadam identities with binomial coefficients

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Abstract

In this note, we introduce a very simple approach to obtain Horadam identities with binomial coefficients including an additional parameter. Many known Fibonacci identities (as well as polynomial identities) will follow immediately as special cases.

Keywords: Horadam number, Fibonacci number, binomial transform

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1. Introduction and motivation

Layman [15] recalled the Fibonacci identities

$$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k, \quad 2^n F_n = \sum_{k=0}^{n} \binom{n}{k} F_{2k}, \quad 3^n F_n = \sum_{k=0}^{n} \binom{n}{k} F_{4k},$$

and attributed them to Hoggatt [9]. Here, as usual, F_n is the nth Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n, n \geq 0$. Layman proved more such identities, in particular, the following alternating sums:

$$(-1)^n F_{3n} = \sum_{k=0}^{n} \binom{n}{k} (-2)^k F_{2k}, \quad (-5)^n F_{3n} = \sum_{k=0}^{n} \binom{n}{k} (-2)^k F_{5k},$$

*Statements and conclusions made in this article are entirely those of the author. They do not necessarily reflect the views of LBBW.

and

$$(-4)^n F_{3n} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{6k}.$$

Several additional sums of this kind and generalizations were derived by Carlitz and Ferns [5], Carlitz [4], Haukkanen [7, 8] and Prodinger [16]. More recently, some authors also worked on generalizations and derived expressions for sums with weighted binomial sums, sums with polynomials and sums where only half of the binomial coefficients are used. We refer to [2] and [10–14]. Adegoke [1] generalized many of the above results and derived summation identities involving Horadam numbers and binomial coefficients, some of which we will encounter below.

In this note, we give another type of generalization of some Horadam binomial sums. More precisely, we introduce a very simple approach to obtain Horadam identities with an additional parameter. All results are derived completely routinely using standard methods.

Let $w_n = w_n(a, b; p, q)$ be a general Horadam sequence, i.e., a second order recurrence

$$w_n = pw_{n-1} - qw_{n-2}, \quad n \geq 2,$$

with nonzero constant p, q and initial values $w_0 = a$, $w_1 = b$. We mention the following instances: $w_n(0, 1; 1, -1) = F_n$ is the Fibonacci sequence, $w_n(0, 1; 2, -1) = P_n$ is the Pell sequence, $w_n(0, 1; 1, -2) = J_n$ is the Jacobsthal sequence, $w_n(0, 1; 3, 2) = M_n$ is the Mersenne sequence, $w_n(0, 1; 6, 1) = B_n$ is the balancing number sequence, $w_n(2, 1; 1, -1) = L_n$ is the Lucas sequence, $w_n(2, 2; 2, -1) = Q_n$ is the Pell-Lucas sequence, $w_n(2, 1; 1, -2) = j_n$ is the Jacobsthal-Lucas sequence, and $w_n(1, 3; 6, 1) = C_n$ is Lucas-balancing number sequence. All sequences are listed in OEIS [17] where additional information and references are available. We also note that the sequence w_n also contains important sequences of polynomials: $w_n(0, 1; x, -1) = F_n(x)$ are the Fibonacci polynomials, $w_n(0, 1; 2x, -1) = P_n(x)$ are the Pell polynomials, $w_n(0, 1; 1, -2x) = J_n(x)$ are the Jacobsthal polynomials, and $w_n(0, 1; 6x, 1) = B_n(x)$ are the balancing polynomials, respectively.

The Binet formula of w_n in the non-degenerated case, $p^2 - 4q > 0$, is

$$w_n = A\alpha^n + B\beta^n,$$

with

$$A = \frac{b - a\beta}{\alpha - \beta}, \quad B = \frac{a\alpha - b}{\alpha - \beta},$$

and where α and β are roots of the equation $x^2 - px + q = 0$, that is

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2}, \quad \beta = \frac{p - \sqrt{p^2 - 4q}}{2}.$$

In what follows we will need the following expressions:

$$\alpha + \beta = p, \quad \alpha\beta = q, \quad \alpha - \beta = \sqrt{p^2 - 4q},$$

as well as

$$\alpha^2 = p\alpha - q, \quad \beta^2 = p\beta - q, \tag{1.1}$$

$$\alpha^3 = (p^2 - q)\alpha - pq, \quad \beta^2 = (p^2 - q)\beta - pq, \tag{1.2}$$

$$\alpha^4 = (p^3 - 2pq)\alpha - q(p^2 - q), \quad \beta^2 = (p^3 - 2pq)\beta - q(p^2 - q),$$

and so on.

Finally, we mention the standard fact about sequences and their binomial transforms [3]: Let $(a_n)_{n\geq 0}$ be a sequence of numbers and $(b_n)_{n\geq 0}$ be its binomial transform. Then, we have the following relations:

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \quad \Leftrightarrow \quad a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k.$$

2. A simple generalization

The next lemma will be the key ingredient to derive our results.

Lemma 2.1. *Let n and j be integers with $0 \leq j \leq n$. Then, for each $a, x \in \mathbb{C}$ we have the identity*

$$\binom{n}{j} x^j (a \pm x)^{n-j} = \sum_{k=j}^{n} \binom{k}{j} \binom{n}{k} (\pm 1)^{k-j} x^k a^{n-k}.$$

Proof. From the binomial theorem we have

$$\begin{aligned}
\binom{n}{j} x^j (a \pm x)^{n-j} &= \binom{n}{j} \sum_{m=0}^{n-j} \binom{n-j}{m} (\pm 1)^m x^{m+j} a^{n-(j+m)} \\
&= \binom{n}{j} \sum_{k=j}^{n} \binom{n-j}{k-j} (\pm 1)^{k-j} x^k a^{n-k} \\
&= \sum_{k=j}^{n} \binom{k}{j} \binom{n}{k} (\pm 1)^{k-j} x^k a^{n-k},
\end{aligned}$$

where in the last step we have used the identity

$$\binom{n}{j} \binom{n-j}{k-j} = \binom{n}{k} \binom{k}{j}, \quad 0 \leq j \leq k \leq n. \qquad \square$$

Example 2.2. Setting $(x; a) = (\alpha^2; -q)$, $(x; a) = (\beta^2; -q)$, using (1.1) and the linearity of the Binet form gives the following identity valid for all $0 \leq j \leq n$

$$\binom{n}{j} p^{n-j} w_{n+j+m} = \sum_{k=j}^{n} \binom{k}{j} \binom{n}{k} q^{n-k} w_{2k+m}, \quad m \geq 0.$$

The case $j = 0$ and the corresponding inverse binomial transform produce immediately

$$\left(\frac{p}{q}\right)^n w_{n+m} = \sum_{k=0}^{n} \binom{n}{k} q^{-k} w_{2k+m}$$

as well as

$$q^{-n} w_{2n+m} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left(\frac{p}{q}\right)^k w_{k+m}.$$

Obviously, with $w_n = F_n$ (or L_n) we recover the classical results, which appeared in [5] and [15]. The balancing number counterparts were stated in [6].

Example 2.3. If we set $\Delta = p^2 - 4q$, then a simple computation shows that

$$\alpha^2 - q = \sqrt{\Delta}\alpha \quad \text{and} \quad \beta^2 - q = -\sqrt{\Delta}\alpha.$$

Thus, with $(x; a) = (\alpha^2; -q)$, $(x; a) = (\beta^2; -q)$ and using again (1.1), we see that if n and j have the same parity, then for all $0 \le j \le n$

$$\binom{n}{j} \Delta^{(n-j)/2} w_{n+j+m} = \sum_{k=j}^{n} \binom{k}{j} \binom{n}{k} (-q)^{n-k} w_{2k+m}, \quad m \ge 0.$$

Especially, for $j = 0$ and n even we get

$$q^{-n} \Delta^{n/2} w_{n+m} = \sum_{k=0}^{n} \binom{n}{k} (-q)^{-k} w_{2k+m}$$

and for $j = 1$ and n odd

$$-q^{-n} \Delta^{(n-1)/2} n w_{n+1+m} = \sum_{k=1}^{n} \binom{n}{k} k (-q)^{-k} w_{2k+m}.$$

If n and j are of unequal parity (n odd and j even, for instance), then

$$\binom{n}{j} \Delta^{(n-1-j)/2} v_{n+j+m} = \sum_{k=j}^{n} \binom{k}{j} \binom{n}{k} (-q)^{n-k} u_{2k+m}, \quad m \ge 0,$$

and

$$\binom{n}{j} \Delta^{(n+1-j)/2} u_{n+j+m} = \sum_{k=j}^{n} \binom{k}{j} \binom{n}{k} (-q)^{n-k} v_{2k+m}, \quad m \ge 0,$$

with $u_n = w_n(0, 1; p, q)$ and $v_n = w_n(2, p; p, q)$.

Example 2.4. Setting $(x; a) = (\alpha^3; -pq)$, $(x; a) = (\beta^3; -pq)$, using (1.2) yields for all $0 \le j \le n$

$$\binom{n}{j} (p^2 - q)^{n-j} w_{n+2j+m} = \sum_{k=j}^{n} \binom{k}{j} \binom{n}{k} (pq)^{n-k} w_{3k+m}, \quad m \ge 0.$$

The case $j=0$ in combination with the binomial transform produce

$$\Big(\frac{p^2-q}{pq}\Big)^n w_{n+m} = \sum_{k=0}^{n} \binom{n}{k} (pq)^{-k} w_{3k+m}$$

as well as

$$(pq)^{-n} w_{3n+m} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \Big(\frac{p^2-q}{pq}\Big)^k w_{k+m}.$$

Example 2.5. Combining the values $(x;a)=(p\alpha^3;q^2)$ and $(x;a)=(p\beta^3;q^2)$ with

$$p\alpha^3+q^2=(p^2-q)\alpha^2, \quad \text{and} \quad p\beta^3+q^2=(p^2-q)\beta^2,$$

Lemma 2.1 gives

$$\binom{n}{j} p^j (p^2-q)^{n-j} w_{2n+j+m} = \sum_{k=j}^{n} \binom{k}{j}\binom{n}{k} p^k q^{2n-2k} w_{3k+m}, \quad m\geq 0.$$

Again, from the case $j=0$ and the binomial transform we get

$$\Big(\frac{p^2-q}{q^2}\Big)^n w_{2n+m} = \sum_{k=0}^{n} \binom{n}{k} \Big(\frac{p}{q^2}\Big)^k w_{3k+m}$$

as well as

$$\Big(\frac{p}{q^2}\Big)^n w_{3n+m} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \Big(\frac{p^2-q}{q^2}\Big)^k w_{2k+m}.$$

Example 2.6. In this example we combine the values $(x;a)=(\alpha^4;q(p^2-q))$ and $(x;a)=(\beta^4;q(p^2-q))$ to get

$$\binom{n}{j} (p(p^2-2q))^{n-j} w_{n+3j+m} = \sum_{k=j}^{n} \binom{k}{j}\binom{n}{k} (q(p^2-q))^{n-k} w_{4k+m}, \quad m\geq 0.$$

Hence,

$$\Big(\frac{p(p^2-2q)}{q(p^2-q)}\Big)^n w_{n+m} = \sum_{k=0}^{n} \binom{n}{k} (q(p^2-q))^{-k} w_{4k+m}$$

as well as

$$(q(p^2-q))^{-n} w_{4n+m} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \Big(\frac{p(p^2-2q)}{q(p^2-q)}\Big)^k w_{k+m}.$$

Example 2.7. An application of Lemma 2.1 with $(x;a)=(\alpha^4;q^2)$ and $(x;a)=(\beta^4;q^2)$ and noting that

$$\alpha^4+q^2=(p^2-2q)\alpha^2 \quad \text{and} \quad \beta^4+q^2=(p^2-2q)\beta^2,$$

proves the next identity:

$$\binom{n}{j}(p^2-2q)^{n-j}w_{2n+2j+m} = \sum_{k=j}^{n}\binom{k}{j}\binom{n}{k}q^{2n-2k}w_{4k+m}, \quad m \geq 0.$$

The case $j=0$ in conjunction with the binomial transform yield

$$\Big(\frac{p^2-2q}{q^2}\Big)^n w_{2n+m} = \sum_{k=0}^{n}\binom{n}{k}q^{-2k}w_{4k+m}$$

as well as

$$q^{-2n}w_{4n+m} = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\Big(\frac{p^2-2q}{q^2}\Big)^k w_{2k+m}.$$

3. Slightly more general identities

Lemma 3.1. *For each $n \geq 1$ we have the relations*

$$\alpha^n = \alpha u_n - q u_{n-1} \quad \text{and} \quad \beta^n = \beta u_n - q u_{n-1}$$

with $u_n = w_n(0,1;p,q)$.

Proof. We can prove the statements by induction on n. Since, $\alpha^1 = \alpha u_1 - q u_0$, the inductive step is

$$\begin{aligned}
\alpha^{n+1} &= \alpha\alpha^n \\
&= \alpha^2 u_n - q\alpha u_{n-1} \\
&= (\alpha u_2 - q u_1)u_n - q\alpha u_{n-1} \\
&= \alpha(p u_n - q u_{n-1}) - q u_n \quad (u_2 = p) \\
&= \alpha u_{n+1} - q u_n.
\end{aligned}$$

The proof of the second statement is a copy of the first proof. □

The next identity is stated as a proposition.

Proposition 3.2. *For integers $m \geq 2$, $r \geq 0$ and $0 \leq j \leq n$ it is true that*

$$\binom{n}{j}u_{m-1}^{-n}u_m^{n-j}w_{j(m-1)+n+r} = \sum_{k=j}^{n}\binom{k}{j}\binom{n}{k}u_{m-1}^{-k}q^{n-k}w_{mk+r}.$$

Proof. From Lemma 3.1 we see that

$$q + \frac{\alpha^n}{u_{n-1}} = \alpha\frac{u_n}{u_{n-1}} \quad \text{and} \quad q + \frac{\beta^n}{u_{n-1}} = \beta\frac{u_n}{u_{n-1}}.$$

Using Lemma 2.1 with $a=q$ and $x = \frac{\alpha^n}{u_{n-1}}$ and $x = \frac{\beta^n}{u_{n-1}}$ completes the proof. □

The next two sum identities follow immediately:

$$\Big(\frac{u_m}{qu_{m-1}}\Big)^n w_{n+r} = \sum_{k=0}^{n} \binom{n}{k} (qu_{m-1})^{-k} w_{mk+r}$$

as well as

$$(qu_{m-1})^{-n} w_{mn+r} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \Big(\frac{u_m}{qu_{m-1}}\Big)^k w_{k+r}.$$

We also mention the formula for $j = 1$:

$$nu_{m-1}^{-n} u_m^{n-1} w_{n+m+r-1} = \sum_{k=1}^{n} \binom{n}{k} k u_{m-1}^{-k} q^{n-k} w_{mk+r}.$$

Lemma 3.3. *For each $k, n \geq 1$ we have the relations*

$$\alpha^{kn} = \frac{u_{kn}}{u_n}\alpha^n - q^n \frac{u_{(k-1)n}}{u_n} \quad \text{and} \quad \beta^{kn} = \frac{u_{kn}}{u_n}\beta^n - q^n \frac{u_{(k-1)n}}{u_n}$$

with $u_n = w_n(0, 1; p, q)$.

Proof. Both statements can be verified directly by computation working with $u_n\alpha^{kn}$ (respectively $u_n\beta^{kn}$) and $q = \alpha\beta$. □

Proposition 3.4. *For integers $m \geq 2$, $s \geq 1$, $r \geq 0$, and $0 \leq j \leq n$ we have the identity*

$$\binom{n}{j} \Big(\frac{u_s}{u_{ms}}\Big)^j \Big(\frac{u_{ms}}{u_{(m-1)s}}\Big)^n q^{-sn} w_{sn+sj(m-1)+r}$$
$$= \sum_{k=j}^{n} \binom{k}{j} \binom{n}{k} q^{-sk} \Big(\frac{u_s}{u_{(m-1)s}}\Big)^k w_{msk+r}.$$

Proof. The identity follows upon combining Lemma 2.1 with Lemma 3.3 with $a = q^s u_{(m-1)s}/u_s$ and $x = \alpha^{ms}$ and $x = \beta^{ms}$, respectively. □

The special identities for $j = 0$ are

$$\Big(\frac{u_{ms}}{u_{(m-1)s}}\Big)^n q^{-sn} w_{sn+r} = \sum_{k=0}^{n} \binom{n}{k} q^{-sk} \Big(\frac{u_s}{u_{(m-1)s}}\Big)^k w_{msk+r}$$

and

$$\Big(\frac{u_s}{u_{(m-1)s}}\Big)^n q^{-sn} w_{msn+r} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} q^{-sk} \Big(\frac{u_{ms}}{u_{(m-1)s}}\Big)^k w_{sk+r}.$$

Corollary 3.5. *For integers $s \geq 1$, $r \geq 0$ and $0 \leq j \leq n$ we have the identity*

$$\binom{n}{j} v_s^{n-j} q^{-sn} w_{s(n+j)+r} = \sum_{k=j}^{n} \binom{k}{j}\binom{n}{k} q^{-sk} w_{2sk+r}.$$

In particular,

$$(-1)^n q^{-sn} w_{2sn+r} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{-sk} v_s^k w_{sk+r}$$

and

$$q^{-sn} v_s^n w_{sn+r} = \sum_{k=0}^{n} \binom{n}{k} q^{-sk} w_{2sk+r}.$$

Proof. Set $m = 2$ in Proposition 3.4 and use $u_{2n}/u_n = v_n$. □

Some more examples could be stated, but we stop here, as the principle is clear.

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